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Real interpolation over the real affine plane

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We recall some facts concerning classical interpolation (according to **Newton, Lagrange,** and others) from $\mathbb{R}$ into $\mathbb{R}$ by polynomials. For $0 < n \in \mathbb{Z}$ take different real numbers $z_1, \ldots, z_n$ and arbitrary real numbers $h_1, \ldots, h_n$; let

$$g_j(z) = g_j(z; z_1, \ldots, z_n) := \prod_{\substack{0<k\leq n \\ k\neq j}} \frac{z_k - z}{z_k - z_j} \qquad (0 < j \leq n,\ z \in \mathbb{R}),$$

$$f(z) = f(z; {z_1, \ldots, z_n \atop h_1, \ldots, h_n}) := \sum_{0<j\leq n} h_j g_j(z) \qquad (z \in \mathbb{R}).$$

We mention 4 remarkable facts:

1.) We have

$$f(z_j) = h_j \qquad (0 < j \leq n).$$

2.) We have

$$\sum_{0<j\leq n} g_j(z; z_1, \ldots, z_n) = 1 \qquad (z \in \mathbb{R}) \tag{1}$$

("partition of unity") with the consequence

$$f(z; {z_1, \ldots, z_n \atop h_1+h, \ldots, h_n+h}) = f(z; {z_1, \ldots, z_n \atop h_1, \ldots, h_n}) + h \qquad (z \in \mathbb{R},\ h \in \mathbb{R}).$$

3.) For every invertible affine map ψ: $\mathbb{R} \to \mathbb{R}$ we have

$$f(\psi z; {\psi z_1, \ldots, \psi z_n \atop h_1, \ldots, h_n}) = f(z; {z_1, \ldots, z_n \atop h_1, \ldots, h_n}) \qquad (z \in \mathbb{R}).$$

4.) For every permutation π of $\{1, \ldots, n\}$ we have

$$f(z; {z_{\pi(1)}, \ldots, z_{\pi(n)} \atop h_{\pi(1)}, \ldots, h_{\pi(n)}}) = f(z; {z_1, \ldots, z_n \atop h_1, \ldots, h_n}) \qquad (z \in \mathbb{R}).$$

Denote by A the affine plane over $\mathbb{R}$. In this paper we study interpolation from A into $\mathbb{R}$ by polynomials. The functions $F_{ij}^k : A \to \mathbb{R}$ below generalize the fundamental polynomials $g_j : \mathbb{R} \to \mathbb{R}$.

Let $3 \leq n \in \mathbb{Z}$ and take $P_1,\dots,P_n$ in A. We do not want patterns like ∴∴. or ⁞⁞⁞ for the P_j, since this is too closely related to applying interpolation from $\mathbb{R}$ into $\mathbb{R}$ twice. On the contrary, we suppose:

(K_n) no triple of the P_j is collinear,
(T_n) no quadruple of the P_j forms a trapezoid.

If (K_n) and (T_n) are already satisfied then (K_{n+1}) resp. (T_{n+1}) means that $P_{n+1} \in A$ has to avoid $\binom{n}{2}$ resp. $\binom{n}{2}(n-2)$ lines. For integers i,j,m let $0 < i < j \leq n$, $0 < m \leq n$; denote by G_{ij} the line through P_i and P_j; denote by G^m_{ij} the line through P_m and parallel to G_{ij}. We have

(2) $\quad G^i_{ij} = G^j_{ij} = G_{ij}$.

For $P \in A$ denote by $L_{ij}(P) = 0$ a linear equation of G_{ij}. For $k \in \mathbb{Z}$, $0 < k \leq n$, $k \neq i$, $k \neq j$ we define the function $F^k_{ij} : A \to \mathbb{R}$ by[1])

$$F^k_{ij}(P) = F^k_{ij}(P; P_1,\dots,P_n) := \prod_{\substack{0<h\leq n\\ h\neq i\\ h\neq k}} \frac{L_{ij}(P_h) - L_{ij}(P)}{L_{ij}(P_h) - L_{ij}(P_k)} \qquad (P \in A);$$

if $h=j$ resp. $h\neq j$ then $L_{ij}(P_h) = L_{ij}(P_k)$ would contradict (K_n) resp. (T_n). We have exactly $\binom{n}{2}$ (n–2) functions F^k_{ij}. We have

(3) $$F^k_{ij}(P_m) = \begin{cases} 1 & \text{in case } m=k \\ 0 & \text{otherwise.} \end{cases}$$

For every invertible affine map $\varphi: A \to A$ we have

(4) $$F^k_{ij}(\varphi P; \varphi P_1,\dots,\varphi P_n) = F^k_{ij}(P; P_1,\dots,P_n) \qquad (P \in A).$$

Denote by H_{ij} an arbitrary line in A which is not parallel to G_{ij} and denote by t an arbitrary affine coordinate on H_{ij}. The restriction of F^k_{ij} to H_{ij} is a real polynomial $\tilde{F}^k_{ij}$ in t of degree n–2. From $\tilde{F}^k_{ij}$ we may return to F^k_{ij} by constant continuation along all lines parallel to G_{ij}. In the point P^*_m of intersection of G^m_{ij} and H_{ij} we have

(5) $$\tilde{F}^k_{ij}(P^*_m) = \begin{cases} 1 & \text{in case } m=k \\ 0 & \text{otherwise.} \end{cases}$$

By (2) we may take $m\neq i$ and this gives exactly n–1 points P^*_m; the degree n–2 and (5) determine the polynomial $\tilde{F}^k_{ij}$ uniquely.

For $0 < i < j \leq n$ we define the functions $F_{ij}: A \to \mathbb{R}$ by

[1]) These functions and their generalization to higher dimensions have been mentioned already in [1].

$$(6) \qquad F_{ij}(P) = F_{ij}(P; P_1, \ldots, P_n) := \prod_{\substack{0<h\leq n \\ h\neq i \\ h\neq j}} \frac{L_{ij}(P_h) - L_{ij}(P)}{L_{ij}(P_h)} \qquad (P \in A);$$

let $F_{ji}(P) := F_{ij}(P)$ $(P \in A)$; by (K_n) we have $L_{ij}(P_h) \neq 0$. For every invertible affine map $\varphi: A \to A$ we have

$$(7) \qquad F_{ij}(\varphi P; \varphi P_1, \ldots, \varphi P_n) = F_{ij}(P; P_1, \ldots, P_n) \qquad (P \in A).$$

We have exactly $\binom{n}{2}$ functions F_{ij}. We have

$$(8) \qquad F_{ij}(P_m) = \begin{cases} 1 & \text{in case } m=i \\ 1 & \text{in case } m=j \\ 0 & \text{otherwise.} \end{cases}$$

The restriction of F_{ij} to H_{ij} is a real polynomial $\tilde{F}_{ij}$ in t of degree n–2. From $\tilde{F}_{ij}$ we may return to f_{ij} by constant continuation along all lines parallel to G_{ij}. Let $m \neq i$; we have

$$(9) \qquad \tilde{F}_{ij}(P_m^*) = \begin{cases} 1 & \text{in case } m=j \\ 0 & \text{otherwise.} \end{cases}$$

The degree n–2 and (9) determine the polynomial $\tilde{F}_{ij}$ uniquely.

Lemma 1. *For* $0 < i < j \leq n$ *we have*

$$F_{ij}(P) + \sum_{\substack{0<k\leq n \\ k\neq i \\ k\neq j}} F_{ij}^k(P) = 1 \qquad (P \in A).$$

Proof. For $P \in H_{ij}$ this follows immediately from (9) and (5). Now we observe the twice mentioned constant continuation.

By (3), (8) we may use the F_{ij}^k but not the F_{ij} for interpolation. For $B_m \in \mathbb{R}$ $(0 < m \leq n)$ we define the function $E: A \to \mathbb{R}$ by

$$(10) \qquad E(P) = E\left(P; \begin{matrix} P_1, \ldots, P_n \\ B_1, \ldots, B_n \end{matrix}\right) := \frac{1}{\binom{n-1}{2}} \sum\sum\sum_{\substack{0<i<j\leq n \\ 0<k\leq n \\ k\neq i, k\neq j}} B_k F_{ij}^k(P) \qquad (P \in A).$$

(10) implies

$$E(P_m) = B_m \qquad (0 < m \leq n).$$

For every invertible affine map $\varphi: A \to A$ we have

$$E\left(\varphi P; \begin{matrix} \varphi P_1, \ldots, \varphi P_n \\ B_1, \ldots, B_n \end{matrix}\right) = E\left(P; \begin{matrix} P_1, \ldots, P_n \\ B_1, \ldots, B_n \end{matrix}\right) \qquad (P \in A)$$

by (4). For every permutation π of $\{1,\ldots,n\}$ we have obviously

$$E(P;\ {P_{\pi(1)},\ldots,P_{\pi(n)} \atop B_{\pi(1)},\ldots,B_{\pi(n)}}) = E(P;\ {P_1,\ldots,P_n \atop B_1,\ldots,B_n}) \qquad (P \in A).$$

Let $B \in \mathbb{R}$; we would like to show

$$E(P;\ {P_1,\ldots,P_n \atop B_1+B,\ldots,B_n+B}) = E(P;\ {P_1,\ldots,P_n \atop B_1,\ldots,B_n}) + B \qquad (P \in A);$$

for this, obviously

$$(11) \qquad \sum_{\substack{0<i<j\leq n \\ 0<k\leq n \\ k\neq i,\, k\neq j}} \sum \sum F_{ij}^{k}(P) = \binom{n-1}{2} \qquad (P \in A)$$

is sufficient; by Lemma 1, for this it is sufficient to show

Theorem 1. *Let* $n \in \mathbb{Z}$, $n \geq 3$; *for* $P_j \in A$ $(0 < j \leq n)$ *we suppose* (K_n), (T_n); *let*

$$(12) \qquad S = S(P; P_1,\ldots,P_n) := \sum_{0<i<j\leq n} \sum F_{ij}(P; P_1,\ldots,P_n);$$

then we have

$$(13) \qquad S(P; P_1,\ldots,P_n) = n-1 \qquad (P \in A).$$

Remark 1. In (11), (13) it is only essential that on the right hand sides we have a number depending on n only; its value then is obvious by (3), (8) putting $P = P_1$.

Remark 2. The analogue (1) of (11) was easy to see by looking at the degree and at $z = z_j$ $(0 < j \leq n)$. Here we need more effort.

Remark 3. For $A = \mathbb{R}^2$, $n = 3$ the graphs of F_{12}, F_{13}, F_{23}, E are planes in $\mathbb{R}^3$ and, by (8), Theorem 1 is obvious.

For every invertible affine map $\varphi: A \to A$ we have

$$(14) \qquad S(\varphi P; \varphi P_1,\ldots,\varphi P_n) = S(P; P_1,\ldots,P_n),$$

by (7).

For the proof of Theorem 1 we need some preparations. We may suppose $n > 3$.

We choose an arbitrary affine coordinate system of A; let

$$(15) \qquad P_j = (\xi_j, \eta_j) \quad (0 < j \leq n), \quad P = (\xi, \eta).$$

Then S is a polynomial in ξ, η and a rational function in ξ_j, η_j $(0 < j \leq n)$; S is especially a real rational function in these 2n+2 variables.

We normalize L_{ij} by

$$(16) \qquad L_{ij}(P) = \det \begin{pmatrix} \xi_i & \xi_j & \xi \\ \eta_i & \eta_j & \eta \\ 1 & 1 & 1 \end{pmatrix} \qquad (P \in A).$$

Lemma 2. *All non-constant factors of all denominators of S can be removed.*

Proof. We may look for example at $L_{12}(P_3)$. This means we have to look at those summands of S where $L_{12}(P_3)$ or $L_{13}(P_2)$ or $L_{23}(P_1)$ appears in a denominator. We observe $L_{12}(P_3) = -L_{13}(P_2) = L_{23}(P_1)$. Hence by (6) it is sufficient to prove that

$$(17)\qquad \begin{aligned} &(L_{12}(P_3)-L_{12}(P)) \prod_{3<h\leq n} \frac{L_{12}(P_h)-L_{12}(P)}{L_{12}(P_h)} \\ &-(L_{13}(P_2)-L_{13}(P)) \prod_{3<h\leq n} \frac{L_{13}(P_h)-L_{13}(P)}{L_{13}(P_h)} \\ &+(L_{23}(P_1)-L_{23}(P)) \prod_{3<h\leq n} \frac{L_{23}(P_h)-L_{23}(P)}{L_{23}(P_h)} \end{aligned}$$

has the factor $L_{12}(P_3)$. To make calculations simple we choose an affine coordinate system of A with $P_1=(0,0)$, $P_2=(1,0)$; then $L_{12}(P_3)=\eta_3$. On the one hand we have

$$\begin{aligned} \frac{L_{12}(P_h)-L_{12}(P)}{L_{12}(P_h)} &= \frac{\eta_h-\eta}{\eta_h}, \\ \frac{L_{13}(P_h)-L_{13}(P)}{L_{13}(P_h)} &= \frac{\eta_h\xi_3-\xi_h\eta_3-\eta\xi_3+\xi\eta_3}{\eta_h\xi_3-\xi_h\eta_3}, \\ \frac{L_{23}(P_h)-L_{23}(P)}{L_{23}(P_h)} &= \frac{\eta_h(\xi_3-1)+(1-\xi_h)\eta_3+\eta(1-\xi_3)+(\xi-1)\eta_3}{\eta_h(\xi_3-1)+(1-\xi_h)\eta_3}; \end{aligned}$$

the differences of these fractions have the factor η_3. On the other hand we have

$$(L_{12}(P_3)-L_{12}(P))-(L_{13}(P_2)-L_{13}(P))+(L_{23}(P_1)-L_{23}(P)) = 2\eta_3.$$

Substitution into (17) gives the result.

Let $P_1,\dots,P_n$ in A be given, satisfying (K_n) and (T_n). We choose an affine coordinate system C of A such that none of the 2 coordinate axes is parallel to G_{ij} $(0<i<j\leq n)$ and such that $\xi_i\neq 0$, $\eta_i\neq 0$ $(0<i\leq n)$. By (12) and Lemma 2, S has the form

$$(18)\qquad S(P;P_1,\dots,P_n) = \underset{\text{(finite)}}{\sum\dots\sum} a_{r_1s_1\dots r_ns_nrs}(C)\,\xi_1^{r_1}\eta_1^{s_1}\dots\xi_n^{r_n}\eta_n^{s_n}\xi^r\eta^s$$

with real $a\dots(C)$. We now replace $P_n=(\xi_n,\eta_n)$ by $P_n'=(\xi_n,t)$ with variable t. Except for $\binom{n-1}{2}$ resp. $\binom{n-1}{2}(n-3)$ values of t, the points $P_1,\dots,P_{n-1},P_n'$ satisfy (K_n) resp. (T_n). What happens to $S(P;P_1,\dots,P_{n-1},P_n')$ as $t\to\infty$? By (12) and (6), only those fractions

$$(19)\qquad \frac{L_{ij}(P_h)-L_{ij}(P)}{L_{ij}(P_h)}$$

are effected with $j=n$ resp. $h=n$; in (19) we then have

$$\rightarrow \frac{\xi_h - \xi}{\xi_h - \xi_i} \text{ resp.} \rightarrow 1;$$

consequently, $S(P; P_1, \ldots, P_{n-1}, P'_n)$ is bounded as $t \rightarrow \infty$. This means that terms with η_n do not show up in (18). This argument for η_n is now repeated for $\xi_n, \eta_{n-1}, \ldots, \eta_1, \xi_1$ and we find

$$(20) \qquad S(P; P_1, \ldots, P_n) = \underset{\text{(finite)}}{\sum\sum} a_{rs}(C)\, \xi^r \eta^s$$

with real $a_{rs}(C)$.

Proof of Theorem 1. The assertion follows immediately from (20), (14) for translations φ, and Remark 1.

Take a convex region $M \subset A$ and a function β: $M \rightarrow \mathbb{R}$ with certain differentiability properties; let $P_j \in M$, $B_j := \beta(P_j)$ $(0 < j \leq n)$. The approximation of β on M by (10) is usually not very good since on G_{12} for instance we can only rely on $E(P_1) = \beta(P_1)$, $E(P_2) = \beta(P_2)$. But in (10) we may use limits (like $P_3 \rightarrow G_{12}$) and cover situations more general than (K_n); we shall not discuss this here.

The functions g_j are elegant; for applied interpolation they are not very useful. The same is true for the functions F^k_{ij}.

Example $n = 4$. For A we take $\mathbb{R}^2$ and let $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (0,1)$, $P_4 = (u,v)$ with real u and v, and $P = (x,y)$. (K_4) resp. (T_4) reads $uv(1-u-v) \neq 0$ resp. $(1-u)(1-v)(u+v) \neq 0$. We have

$$F^3_{12}(x,y) = \frac{y}{1}\,\frac{y-v}{1-v},\ F^4_{12}(x,y) = \frac{y}{v}\,\frac{y-1}{v-1},\ F^2_{13}(x,y) = \frac{x}{1}\,\frac{x-u}{1-u},$$

$$F^4_{13}(x,y) = \frac{x}{u}\,\frac{x-1}{u-1},\ F^2_{14}(x,y) = \frac{vx-uy}{v}\,\frac{vx+u(1-y)}{v+u},\ F^3_{14}(x,y) = \frac{vx-uy}{u}\,\frac{v(x-1)-uy}{v+u},$$

$$F^1_{23}(x,y) = \frac{x+y-1}{1}\,\frac{x+y-u-v}{u+v},\ F^4_{23}(x,y) = \frac{x+y-1}{u+v-1}\,\frac{x+y}{u+v},$$

$$F^1_{24}(x,y) = \frac{v(x-1)+(1-u)y}{v}\,\frac{vx+(1-u)(y-1)}{1-u},$$

$$F^3_{24}(x,y) = \frac{v(x-1)+(1-u)y}{1-u-v}\,\frac{vx+(1-u)y}{1-u},$$

$$F^1_{34}(x,y) = \frac{(v-1)x+u(1-y)}{u}\,\frac{(v-1)(x-1)-uy}{1-v},$$

$$F^2_{34}(x,y) = \frac{(v-1)x+u(1-y)}{u+v-1}\,\frac{(v-1)x-uy}{v-1}.$$

(K_4) resp. (T_4) is equivalent to the fact that all first resp. second denominators are different from 0. According to (11), these functions add up to 3; the reader is invited to check this directly. We have

$$F_{12}(x,y) = \frac{y-v}{v}(y-1),\ F_{13}(x,y) = \frac{x-u}{u}(x-1),$$

$$F_{14}(x,y) = \frac{vx+u(1-y)}{u}\,\frac{v(1-x)+uy}{v},$$

$$F_{23}(x,y) = \frac{x+y-u-v}{1-u-v}(x+y),$$

$$F_{24}(x,y) = \frac{vx+(1-u)(y-1)}{u+v-1}\,\frac{vx+(1-u)y}{v},$$

$$F_{34}(x,y) = \frac{(1-v)(x-1)+uy}{u+v-1}\,\frac{(1-v)x+uy}{u}.$$

According to (13), these functions add up to 3; the reader is invited to check this directly.

Example n=5. Continuing the preceeding example, let $P_5=(p,q)$ with real p and q. We have

$$F_{12}^{3}(x,y) = \frac{y}{1}\,\frac{y-v}{1-v}\,\frac{y-q}{1-q},\ F_{12}^{4}(x,y) = \frac{y}{v}\,\frac{y-q}{v-q}\,\frac{y-1}{v-1},$$

$$F_{12}^{5}(x,y) = \frac{y}{q}\,\frac{y-1}{q-1}\,\frac{y-v}{q-v},\ F_{13}^{2}(x,y) = \frac{x}{1}\,\frac{x-u}{1-u}\,\frac{x-p}{1-p},$$

$$F_{13}^{4}(x,y) = \frac{x}{u}\,\frac{x-p}{u-p}\,\frac{x-1}{u-1},\ F_{13}^{5}(x,y) = \frac{x}{p}\,\frac{x-1}{p-1}\,\frac{x-u}{p-u},$$

$$F_{14}^{2}(x,y) = \frac{vx-uy}{v}\,\frac{vx+u(1-y)}{v+u}\,\frac{v(x-p)+u(q-y)}{v(1-p)+uq},$$

$$F_{14}^{3}(x,y) = \frac{vx-uy}{u}\,\frac{v(x-p)+u(q-y)}{u(q-1)-vp}\,\frac{v(x-1)-uy}{v+u},$$

$$F_{14}^{5}(x,y) = \frac{vx-uy}{vp-uq}\,\frac{v(x-1)-uy}{v(p-1)-uq}\,\frac{vx+u(1-y)}{vp+u(1-q)},$$

$$F_{15}^{2}(x,y) = \frac{qx-py}{q}\,\frac{qx+p(1-y)}{q+p}\,\frac{q(x-u)+p(v-y)}{q(1-u)+pv},$$

$$F_{15}^{3}(x,y) = \frac{qx-py}{p}\,\frac{q(x-u)+p(v-y)}{p(v-1)-qu}\,\frac{q(x-1)-py}{q+p},$$

$$F_{15}^{4}(x,y) = \frac{qx-py}{qu-pv}\,\frac{q(x-1)-py}{q(u-1)-pv}\,\frac{qx+p(1-y)}{qu+p(1-v)},$$

$$F_{23}^{1}(x,y) = \frac{1-x-y}{1}\,\frac{x+y-u-v}{u+v}\,\frac{x+y-p-q}{p+q},$$

$$F_{23}^{4}(x,y) = \frac{x+y-1}{u+v-1}\,\frac{x+y-p-q}{u+v-p-q}\,\frac{x+y}{u+v},$$

$$F_{23}^5(x,y)=\frac{x+y-1}{p+q-1}\,\frac{x+y}{p+q}\,\frac{x+y-u-v}{p+q-u-v},$$

$$F_{24}^1(x,y)=\frac{v(x-1)+(1-u)y}{v}\,\frac{vx+(1-u)(y-1)}{u-1}\,\frac{v(x-p)+(1-u)(y-q)}{vp+(1-u)q},$$

$$F_{24}^3(x,y)=\frac{v(x-1)+(1-u)y}{1-u-v}\,\frac{v(x-p)+(1-u)(y-q)}{(1-u)(1-q)-vp}\,\frac{vx+(1-u)y}{1-u},$$

$$F_{24}^5(x,y)=\frac{v(x-1)+(1-u)y}{v(p-1)+(1-u)q}\,\frac{vx+(1-u)y}{vp+(1-u)q}\,\frac{vx+(1-u)(y-1)}{vp+(1-u)(q-1)},$$

$$F_{25}^1(x,y)=\frac{q(x-1)+(1-p)y}{q}\,\frac{qx+(1-p)(y-1)}{p-1}\,\frac{q(x-u)+(1-p)(y-v)}{qu+1(1-p)v},$$

$$F_{25}^3(x,y)=\frac{q(x-1)+(1-p)y}{1-p-q}\,\frac{q(x-u)+(1-p)(y-v)}{(1-p)(1-v)-qu}\,\frac{qx+(1-p)y}{1-p},$$

$$F_{25}^4(x,y)=\frac{q(x-1)+(1-p)y}{q(u-1)+(1-p)v}\,\frac{qx+(1-p)y}{qu+(1-p)v}\,\frac{qx+(1-p)(y-1)}{qu+(1-p)(v-1)},$$

$$F_{34}^1(x,y)=\frac{(v-1)x+u(1-y)}{u}\,\frac{(v-1)(x-1)-uy}{1-v}\,\frac{(v-1)(x-p)+u(q-y)}{(1-v)p+uq},$$

$$F_{34}^2(x,y)=\frac{(v-1)x+u(1-y)}{u+v-1}\,\frac{(v-1)(x-p)+u(q-y)}{(v-1)(1-p)+uq}\,\frac{(v-1)x-uy}{v-1},$$

$$F_{34}^5(x,y)=\frac{(v-1)x+u(1-y)}{(v-1)p+u(1-q)}\,\frac{(v-1)x-uy}{(v-1)p-uq}\,\frac{(v-1)(x-1)-uy}{(v-1)(p-1)-uq},$$

$$F_{35}^1(x,y)=\frac{(q-1)x+p(1-y)}{p}\,\frac{(q-1)(x-1)-py}{1-q}\,\frac{(q-1)(x-u)+p(v-y)}{(1-q)u+pv},$$

$$F_{35}^2(x,y)=\frac{(q-1)x+p(1-y)}{p+q-1}\,\frac{(q-1)(x-u)+p(v-y)}{(q-1)(1-u)+pv}\,\frac{(q-1)x-py}{q-1},$$

$$F_{35}^4(x,y)=\frac{(q-1)x+p(1-y)}{(q-1)u+p(1-v)}\,\frac{(q-1)x-py}{(q-1)u-pv}\,\frac{(q-1)(x-1)-py}{(q-1)(u-1)-pv},$$

$$F_{45}^1(x,y)=\frac{(q-v)x+(u-p)y+pv-qu}{pv-qu}\,\frac{(q-v)x+(u-p)y+v-q}{v-q}\,\frac{(q-v)x+(u-p)y+p-u}{p-u},$$

$$F_{45}^2(x,y)=\frac{(q-v)x+(u-p)y+pv-qu}{q(1-u)+v(p-1)}\,\frac{(q-v)x+(u-p)y+p-u}{q-v+p-u}\,\frac{(q-v)x+(u-p)y}{q-v},$$

$$F_{45}^3(x,y)=\frac{(q-v)x+(u-p)y+pv-qu}{(1-q)u+p(v-1)}\,\frac{(q-v)x+(u-p)y}{u-p}\,\frac{(q-v)x+(u-p)y+v-q}{u+v-p-q}.$$

(K_5) resp. (T_5) is equivalent to the fact that all first resp. all the other denominators are different from 0. According to (11), these 30 functions add up to 6; the reader is invited to check this directly. We have

$$F_{12}(x,y) = (1-y)\,\frac{v-y}{v}\,\frac{q-y}{q},\quad F_{13}(x,y) = (1-x)\,\frac{u-x}{u}\,\frac{p-x}{p},$$

$$F_{14}(x,y) = \frac{v(1-x)+uy}{v}\,\frac{vx+u(1-y)}{u}\,\frac{v(p-x)-u(q-y)}{vp-uq},$$

$$F_{15}(x,y) = \frac{q(1-x)+py}{q}\,\frac{qx+p(1-y)}{p}\,\frac{q(u-x)-p(v-y)}{qu-pv},$$

$$F_{23}(x,y) = (x+y)\,\frac{u+v-x-y}{u+v-1}\,\frac{p+q-x-y}{p+q-1},$$

$$F_{24}(x,y) = \frac{vx+(1-u)y}{v}\,\frac{vx+(1-u)(y-1)}{u+v-1}\,\frac{v(p-x)+(1-u)(q-y)}{v(p-1)+(1-u)q},$$

$$F_{25}(x,y) = \frac{qx+(1-p)y}{q}\,\frac{qx+(1-p)(y-1)}{p+q-1}\,\frac{q(u-x)+(1-p)(v-y)}{q(u-1)+(1-p)v},$$

$$F_{34}(x,y) = \frac{(1-v)x+uy}{u}\,\frac{(1-v)(x-1)+uy}{u+v-1}\,\frac{(1-v)(p-x)+u(q-y)}{(1-v)p+u(q-1)},$$

$$F_{35}(x,y) = \frac{(1-q)x+py}{p}\,\frac{(1-q)(x-1)+py}{p+q-1}\,\frac{(1-q)(u-x)+p(v-y)}{(1-q)u+p(v-1)},$$

$$F_{45}(x,y) = \frac{(q-v)x+(u-p)y}{qu-pv}\,\frac{(q-v)(x-1)+(u-p)y}{q(u-1)+(1-p)v}\,\frac{(q-v)x+(u-p)(y-1)}{(1-v)p+u(q-1)}.$$

According to (13), these 10 functions add up to 4; the reader is invited to check this directly.

By the way, we worked the Examples $n=4$, $n=5$ first and looked for a proof of (11) afterwards.

References

[1] G. J. Rieger, On interpolation in 3-dimensional space. Portugaliae Math. **30** (1971), 119–136.